Week of March 2

1 General Tips

- **Always simplify first!!** Before diving headfirst into any problem, you should always try to see if you can simplify the given in any way. For example, can you find common divisors in the numerator and denominator of fractions or rewrite terms so that the given presentation is less terrible? Simplifying things at the start can make computations much easier down the road.

- **Factoring polynomials.** Given polynomials may not always be quadratic polynomials, and even if they are, they may not look easily factorable at first glance. If you cannot use your usual tricks (difference of squares, quadratic formula, etc), then try finding roots by plugging in some nice numbers (e.g. plug in 1 or -1). If the polynomial miraculously evaluates to 0, then your root must show up in the factorization, so your polynomial has to take the form

$$(x - \text{root})(\text{some other polynomial of smaller degree}).$$

- **Products in integrals.** The integral of a product is not the product of an integral. As tempting as it is to write

$$\int u(x)v(x)\,dx = \int u(x)\,dx \cdot \int v(x)\,dx,$$

that is unfortunately not how integrals work. Instead, you can try to use integration by parts to simplify the product within the integral.

- **Partial fractions.** Remember that when the denominator has powers like so:

$$\frac{x}{(x-1)^2(x+2)}$$

so you have to start your partial fractions off with *three* terms like this:

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} = \frac{x}{(x-1)^2(x+2)}$$

The gist of why you need three terms is because having a denominator that is a cubic polynomial means your numerator (after long division) can be a quadratic polynomial. A quadratic polynomial has three terms, so you have (more or less) three degrees of freedom. Another way to think about it is that you want your partial fractions to be of the form

$$\frac{\alpha x + \beta}{(x-1)^2} + \frac{\gamma}{x+2} = \frac{x}{(x-1)^2(x+2)}$$

α would correspond to A and β would correspond to $B - A$ from the previous format. One can see this by adding $A/(x-1)$ and $B/(x-1)^2$ by making the denomiators equal and comparing the coefficients. (Note that this would make $\gamma = C$.)

- **Observing formats.** Many complicated integrals look complicated because they are trying to combine different methods. So you may have to end up doing some combination of techniques to solve the integral. My advice for these types of integrals is to notice patterns. For example, you know that integration by parts problems should look like the product of a differentiable part and an integrable part. Similarly, a u-substitution must constitute of a function and its derivative. Recognizing these patterns/formats is one of the best ways to figure out what methods of integration will result in a fruitful computation.

2 Partial Fractions

- Compute the following integral

$$\int \frac{6x+13}{x^2+5x+6}\,dx.$$

Then compute the definite integral

$$\int_0^3 \frac{6x+13}{x^2+5x+6}\,dx.$$

- Compute the following integral
$$\int \frac{x^3 - 27}{x^2 - 9}\, dx.$$
Then compute the definite integral
$$\int_4^9 \frac{x^3 - 27}{x^2 - 9}\, dx.$$

- Compute the following integral [may get messy]

$$\int \frac{x^5 - 4x + 1}{x^3 - 3x^2 - 33x + 35}\, dx$$

Then compute the definite integral

$$\int_9^1 5\frac{x^5 - 4x + 1}{x^3 - 3x^2 - 33x + 35}\, dx$$

- Compute the following integral [may get messy]

$$\int \frac{x^7 - x^4 + 2x + 1}{x^3 - x^2}\, dx$$

Then compute the definite integral

$$\int_3^7 \frac{x^7 - x^4 + 2x + 1}{x^3 - x^2}\, dx$$

- Compute the following integral [may get messy]

$$\int \frac{x^5 - x^4 + 2x + 1}{x^2 - 7}\,dx$$

- Compute the following integral [may get messy]

$$\int \frac{x^7 - x^4 + 2x + 1}{x^3 - x^2} \, dx$$

3 Integration by parts

- Compute the following integral

$$\int x^2 \ln(x)\, dx$$

then compute the definite integral

$$\int_1^9 x^2 \ln(x)\, dx$$

- Compute the following integral

$$\int_1^3 \ln(2x)\,dx.$$

- Compute the following integral in two different ways

$$\int \frac{\ln(x)}{x}\, dx.$$

[Hint: one method will end up not actually integrating anything after the first step.]

- Compute the following integral in as many different ways as you can

$$\int (2x+1)(x-2)^2 \, dx.$$

- Compute the following integral in as many different ways as you can

$$\int_{-1}^{1} x(x^4 + x^2)^2 \, dx.$$

- Compute the following integral in as many different ways as you can

$$\int_1^e \ln(x^2)\, dx.$$

- Compute the following integral

$$\int x(\ln(x))^3\,dx.$$

- Compute the following integral

$$\int \frac{(\ln(x))^3 + (\ln(x))^2 + 5\ln(x) - 4}{x(\ln(x))^3 + 5x(\ln(x))^2 - \ln(x) - 5}\, dx.$$